## Matrices and Determinants

| Level - 3 | Daily Tutorial Sheet-14 |
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**162.** Putting $s-a=\alpha, s-b=\beta, s-c=\gamma,$ we get

$$a=2s-b-c=(s-b)+(s-c)=\alpha+\beta$$

Similarly, $b=\gamma+\alpha, c=\alpha+\beta$. Also,

$$\alpha+\beta+\gamma=3s-(a+b+c)=3s-2s=s$$

$$\therefore \quad \Delta=\begin{vmatrix}(\beta+\gamma)^2 & \alpha^2 & \alpha^2\\ \beta^2 & (\gamma+\alpha^2) & \beta^2\\ \gamma^2 & \gamma^2 & (\alpha+\beta)^2\end{vmatrix}$$

$$=\begin{vmatrix}(\beta+\gamma)^2 & \alpha^2-(\beta+\gamma)^2 & \alpha^2-(\beta+\gamma)^2\\ \beta^2 & (\gamma+\alpha)^2-\beta^2 & 0\\ \gamma^2 & 0 & (\alpha+\beta)^2-\gamma^2\end{vmatrix}$$

$$[C_2\to C_2-C_1, C_3\to C_3-C_1]$$

$$=(\alpha+\beta+\gamma)^2\times\begin{vmatrix}(\beta+\gamma)^2 & \alpha-\beta-\gamma & \alpha-\beta-\gamma\\ \beta^2 & \gamma+\alpha-\beta & 0\\ \gamma^2 & 0 & \alpha+\beta-\gamma\end{vmatrix}$$

[taking $(\alpha+\beta+\gamma)$ common from $C_2$ and $C_3$]

$$=2(\alpha+\beta+\gamma)^2\times\begin{vmatrix}\beta\gamma & -\gamma & -\beta\\ \beta^2 & \gamma+\alpha-\beta & 0\\ \gamma^2 & 0 & \alpha+\beta-\gamma\end{vmatrix}$$

$$\left[R_1\to R_1-R_2-R_3 \text{ and then } R_1\to\frac{1}{2}R_1\right]$$

Now applying $C_2\to\beta C_2+C_1$ and $C_3\to\gamma C_3+C_1,$ we get

$$\Delta=\frac{2(\alpha+\beta+\gamma)^2}{\beta\gamma}\times\begin{vmatrix}\beta\gamma & 0 & 0\\ \beta^2 & \beta(\gamma+\alpha) & \beta^2\\ \gamma^2 & \gamma^2 & \gamma(\alpha+\beta)\end{vmatrix}$$

$$=\frac{2(\alpha+\beta+\gamma)^2}{\beta\gamma}\beta\gamma\left[(\beta\gamma+\beta\alpha)(\gamma\alpha+\gamma\beta)-\beta^2\gamma^2\right]$$

$$=2(\alpha+\beta+\gamma)^2\left[(\alpha\beta\gamma^2+\beta^2\gamma^2+\alpha^2\beta\gamma+\alpha\beta^2\gamma-\beta^2\gamma^2\right]$$

$$=2(\alpha+\beta+\gamma)^3\alpha\beta\gamma$$

$$=2s^3(s-a)(s-b)(s-c)$$

**163.** Applying $C_1 \to xC_1 + yC_2 + zC_3$ to get the term $\left(x^2+y^2+z^2\right)$

$$\Delta = \frac{1}{x}\begin{vmatrix} a\left(x^2+y^2+z^2\right) & ay+bx & cx+az \\ b\left(x^2+y^2+z^2\right) & by-cz-ax & bz+cy \\ c\left(x^2+y^2+z^2\right) & bz+cy & cz-ax-by \end{vmatrix}$$

Now taking $\left(x^2+y^2+z^2\right)$ common from $C_1$ and then applying $R_1 \to aR_1 + bR_2 + cR_2$ [ to get the term $\left(a^2+b^2+c^2\right)$ ]. We get

$$\Delta = \frac{\left(x^2+y^2+z^2\right)}{ax} \times \begin{vmatrix} \left(a^2+b^2+c^2\right) & y\left(a^2+b^2+c^2\right) & z\left(a^2+b^2+c^2\right) \\ b & by-cz-ax & bz+cy \\ c & bz+cy & cz-ax-by \end{vmatrix}$$

Taking $\left(a^2+b^2+c^2\right)$ common from $R_1$ and then applying $C_2 \to C_2 - yC_1$ and $C_3 \to C_3 - zC_1$ , we get

$$\Delta = \frac{\left(x^2+y^2+z^2\right)\left(a^2+b^2+c^2\right)}{ax} \times \begin{bmatrix} 1 & 0 & 0 \\ b & -cz-ax & cy \\ c & bz & -ax-by \end{bmatrix}$$

Now expanding along $R_1$ we get

$$\Delta = \frac{1}{ax}\left(x^2+y^2+z^2\right)\left(a^2+b^2+c^2\right) \times \left[aczx + bczy + a^2x^2 + abxy - bcyz\right]$$

$$= \left(x^2+y^2+z^2\right)\left(a^2+b^2+c^2\right)\left(ax+by+cz\right)$$

**164.** $\Delta(x) = \begin{vmatrix} a_1+x & b_1+x & c_1+x \\ a_2+x & b_2+x & c_2+x \\ a_3+x & b_3+x & c_3+x \end{vmatrix}$

$$\therefore \quad \Delta'(x) = \begin{vmatrix} 1 & b_1+x & c_1+x \\ 1 & b_2+x & c_2+x \\ 1 & b_3+x & c_3+x \end{vmatrix} + \begin{vmatrix} a_1+x & 1 & c_1+x \\ a_2+x & 1 & c_2+x \\ a_3+x & 1 & c_3+x \end{vmatrix} + \begin{vmatrix} a_1+x & b_1+x & 1 \\ a_2+x & b_2+x & 1 \\ a_3+x & b_3+x & 1 \end{vmatrix}$$

Applying $C_2 \to C_2 - xC_1$ and $C_3 \to C_3 - xC_1$ in the first determinant of R.H.S.,

$C_1 \to C_1 - xC_2$ and $C_3 \to C_3 - xC_2$ in the second determinant, and $C_1 \to C_1 - xC_3$ and $C_2 \to C_2 - xC_3$ in the third determinant, we get $\Delta'(x) = \begin{vmatrix} 1 & b_1 & c_1 \\ 1 & b_2 & c_2 \\ 1 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & 1 & c_1 \\ a_1 & 1 & c_2 \\ a_3 & 1 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$

Now consider the cofactors of $\Delta(0) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Which are $b_2c_3 - b_3c_2, c_2a_3 - c_3a_2, a_2b_3 - b_2a_3$ etc. clearly.

$$\begin{vmatrix} 1 & b_1 & c_1 \\ 1 & b_2 & c_2 \\ 1 & b_3 & c_3 \end{vmatrix} = (b_2c_3 - b_3c_2) + (c_1b_3 - c_3b_1) + (b_1c_2 - b_2c_1)$$

Which is the sum of cofactors of the first row elements of $\Delta(0)$.

Similarly, $\begin{vmatrix} a_1 & 1 & c_1 \\ a_2 & 1 & c_2 \\ a_3 & 1 & c_3 \end{vmatrix}$ and $\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$

Are the sum of cofactors of second row and third row elements, respectively, of $\Delta(0)$. Hence $\Delta'(x) = S$.

where $S$ denotes the sum of all cofactors of elements of $\Delta(0)$. therefore,

$$\Delta''(x) = 0$$

Since $\Delta'(x) = S, \Delta(x) = Sx + k$ so,

$$\Delta(0) = k$$

Hence, $\quad \Delta(x) = xS + \Delta(0)$

**165.** $\alpha, \beta, \gamma$ are roots of $ax^3 + bx^2 + cx + d = 0$

$\therefore \quad ax^3 + bx^2 + cx + d = a(x-\alpha)(x-\beta)(x-\gamma)$

Putting $x = 1$, we get $a + b + c + d = a(1-\alpha)(1-\beta)(1-\gamma)$

Also product of roots $\alpha\beta\gamma = \frac{-d}{a}$

$$\Delta = \begin{vmatrix} \frac{\alpha}{1-\alpha} & \frac{\beta}{1-\beta} & \frac{\gamma}{1-\gamma} \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix}$$

Taking $\alpha, \beta, \gamma$ common from $C_1, C_2, C_3$ respectively, we get

$$\Delta = \alpha\beta\gamma \begin{vmatrix} \frac{1}{1-\alpha} & \frac{1}{1-\beta} & \frac{1}{1-\gamma} \\ 1 & 1 & 1 \\ \alpha & \beta & \gamma \end{vmatrix} = \frac{\alpha\beta\gamma}{(1-\alpha)(1-\beta)(1-\gamma)}$$

$$\begin{vmatrix} (1-\beta)(1-\gamma) & (1-\alpha)(1-\gamma) & (1-\alpha)(1-\beta) \\ 1 & 1 & 1 \\ \alpha & \beta & \gamma \end{vmatrix} = \frac{-\alpha\beta\gamma}{(1-\alpha)(1-\beta)(1-\gamma)}$$

$$\begin{vmatrix} (1-\beta)(1-\gamma) & (1-\alpha)(1-\gamma) & (1-\alpha)(1-\beta) \\ 1 & 1 & 1 \\ 1-\alpha & 1-\beta & 1-\gamma \end{vmatrix} = \frac{-\alpha\beta\gamma}{\left[(1-\alpha)(1-\beta)(1-\gamma)\right]^2}$$

$$\begin{vmatrix} (1-\alpha) & (1-\alpha) & (1-\alpha) \\ \times(1-\beta)(1-\gamma) & \times(1-\beta)(1-\gamma) & \times(1-\beta)(1-\gamma) \\ 1-\alpha & 1-\beta & 1-\gamma \\ (1-\alpha)^2 & (1-\beta)^2 & (1-\gamma)^2 \end{vmatrix} = \frac{-\alpha\beta\gamma}{(1-\alpha)(1-\beta)(1-\gamma)}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1-\alpha & 1-\beta & 1-\gamma \\ (1-\alpha)^2 & (1-\beta)^2 & (1-\gamma)^2 \end{vmatrix} = \frac{-\alpha\beta\gamma(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}{(1-\alpha)(1-\beta)(1-\gamma)} = \frac{\frac{d}{a}\times\frac{25}{2}}{\frac{a+b+c+d}{a}} = \frac{25d}{2(a+b+c+d)}$$

**166.** We have $X^2 = X \times X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O \Rightarrow \quad X^3 = X^2 \times X = O \times X = O$

Similarly, $X^4 = X^5 = X^6 = O$

Now, by binomial theorem, we have

$$(pI + qX)^m = (pI)^m + {}^mC_1 (pI)^{m-1}(qX) + {}^mC_2 (pI)^{m-2}(qX)^2 + .... + {}^mC_m (qX)^m$$

$$= p^m I + mp^{m-1}qX + \frac{m(m-1)}{2!}p^{m-2}(qX)^2 + .... + q^m X^m$$

$$= p^m I + mp^{m-1}qX + O + O + .... + O \quad = p^m + mp^{m-1}qX, \quad \forall p, q \in R$$

**167.** A is upper triangular matrix. So,

$$A = \begin{cases} a_{ij} = 0, & i > j \\ a_{ij} \neq 0, & i \le j \end{cases} \text{ and } A' = \begin{cases} a_{ij} = 0, & i < j \\ a_{ij} \neq 0, & i \ge j \end{cases}$$

B is lower triangular matrix. So,

$$B = \begin{cases} b_{ij} = 0, & i < j \\ b_{ij} \neq 0, & i \ge j \end{cases}$$

And $$B' = \begin{cases} b_{ij} = 0, & i > j \\ b_{ij} \neq 0, & i \le j \end{cases}$$

Let $$C' = A' + B = \begin{cases} c'_{ij} = 0, & i < j \\ c'_{ij} \neq 0, & i \ge j \end{cases}$$

$$C'' = (A + B') = \begin{cases} c''_{ij} = 0, & i > j \\ c''_{ij} \neq 0, & i \le j \end{cases}$$

$\therefore \quad (A' + B) \times (A + B') = C$ (let) for $C, c_{ij} \neq 0$ (for all $i$ and $j$)

Therefore, $(A' + B) \times (A + B')$ is a matrix of order $n \times n, \forall c_{ij} \neq 0$ in any case.